Unsystematic Credit Risk and Coherent Risk Measures

Hans Rau-Bredow
University of Cologne, University of Wuerzburg
E-mail: hans.rau-bredow@mail.uni-wuerzburg.de

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Abstract

The aim of this paper is to combine two hitherto unrelated lines of research, namely the granularity adjustment technique for unsystematic credit risk and the theory of coherent risk measures. In the existing literature, it has always been taken for granted that such a granularity adjustment is positive. In this paper, a counter-example is presented in which the granularity adjustment for *VaR* takes negative value. A coherent risk measure, which is also sub-additive, is given by *ES*. It is shown in this paper that the granularity adjustment technique could also be applied if *ES* is used as a risk measure. It turns out that the granularity adjustment will then be always positive.

Classification words:
Unsystematic Credit Risk, Granularity Adjustment, Coherent Risk Measures, Value at Risk, Expected Shortfall

1. Introduction

The aim of this paper is to combine two hitherto unrelated lines of research, namely the granularity adjustment technique for unsystematic credit risk and the theory of coherent risk measures. Unsystematic or borrower specific credit risk vanishes in a perfectly diversified portfolio. In a conditional world, where the values of the systematic risk factors are taken as given, the aggregated portfolio loss is the sum of stochastic independent random variables. It therefore converges to its conditional mean as more and more loans are added (law of large numbers). The granularity adjustment corrects for the error which is made if risk measures like Value at Risk (*VaR*) or Expected Shortfall *ES* are applied not on the actual portfolio loss variable, but instead on the conditional mean. The technique was introduced by Gordy (2003), (2004) and closed-form expression of the adjustment term have been developed by Wilde (2001), Martin and Wilde (2002) and Emmer/Tasche (2005).

In the existing literature, it has always been taken for granted that such a granularity adjustment is positive. In this paper, a counter-example is presented in which the granularity adjustment for *VaR* takes negative value. If calculation of the capital reserves is based on *VaR*, this would imply a discount in terms of capital reserves for a less diversified credit portfolio. Arztner et al. (1999) have shown that *VaR* is not a coherent measure of risk. In particular, *VaR* is not sub-additive and therefore does not always account correctly for diversification. A coherent risk measure, which is also sub-additive, is given by *ES*. It is shown in this paper that the granularity adjustment technique could also be applied if *ES* is used as a risk measure. It turns out that the granularity adjustment will then be always positive.

The paper is organized as follows. Section 2 develops a general factor model in which the granularity adjustment technique can be embedded. In section 3, the impact of diversification is then analyzed. Granularity adjustments for *VaR* and *ES* are developed in section 4, and some final remarks are given in section 5.

2. The Model

Consider a portfolio of n loans with exposure sizes $A_1,.., A_n$. As a percentage of exposure size, the difference between the current value of each loan and the value at the end of the planning horizon (e.g. one year) is described by a random loss variable L_i. Formally, the relative loss L_i of the value of the loan could be positive as well as negative. It is therefore irrelevant whether losses are defined on a book-value or a mark-to-market basis. For example, if a mark-to-market model is used, an upgrading will result in a gain in market value and consequently implies a negative value of the loss variable L_i.

Let each $L_i = L_i(X,\varepsilon_i)$ be given as a function of some systematic risk factors $X = (X_1,\ldots,X_k)$ and an unsystematic risk factor ε_i. The systematic risk factors may also be called background factors and reflect the state of the business cycle in the different industry sectors. Each systematic risk factor can be thought of being assigned to a certain sector of the economy. The systematic risk factors generally have an influence on more than one borrower in the portfolio and are the reason why default events are stochastic dependent. On the other hand, each unsystematic risk factor ε_i has an influence on only one specific borrower. Unlike for the systematic risk factors, which may or may not be correlated, unsystematic risk factors are always assumed to be pairwise independent.

Many credit risk models can be seen as special cases of this simple but very general approach. Structural models such as the Merton (1974) model or CreditMetrics (1997) assume that default events or rating changes are driven by the evolution of the value of the firm assets, which in turn depend on the realization of some systematic and unsystematic risk factors. The risk factors therefore indirectly determine the potential loss $L_i=L_i(X,\varepsilon_i)$ of each loan. Of course, the concrete functional relationship depends on how the particular model is specified, which however is not relevant for the general analysis.

A well-known example for an intensity or default rate model is CreditRisk+ (1997). This model assumes that default probabilities $p_i = p_i(X)$ are not constant, but a function of certain background factors $X = (X_1,\ldots,X_k)$. In order to match this into the above framework, assume that to each borrower there is assigned an additional unsystematic risk factor ε_i and then define:

$$L_i(X,\varepsilon_i)=\begin{cases} LGD_i, & \text{if } \varepsilon_i < N^{-1}[p_i(X)] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Here, it is assumed that the ε_i are standard normal distributed and N^{-1} is the inverse of the cumulative normal distribution function. LGD_i is the loss given default which will arise with probability $p_i = p_i(X)$.

3. Diversification

Having the general factor model stated, it is now possible to clarify the role of diversification. As a percentage of total exposure, the random loss of the entire portfolio at the end of the risk horizon is

$$L_P = \frac{\sum_{i=1}^{n} A_i L_i}{\sum_{i=1}^{n} A_i} \tag{2}$$

Now assume that the realizations of the systematic risk factors $X = (X_1,...,X_k)$ occur before the realizations of the unsystematic risk factors ε_i. With given values of the systematic risk factors, L_P is the sum of stochastically independent random variables. Thus, the central limit theorem can be applied. Conditional on X, the portfolio loss variable L_P is asymptotically normal-distributed with mean

$$\mu(L_P|X) = \frac{\sum_{i=1}^{n} A_i \mu(L_i|X)}{\sum_{i=1}^{n} A_i} \tag{3}$$

and variance

$$\sigma^2(L_P|X) = \frac{\sum_{i=1}^{n} A_i^2 \sigma^2(L_i|X)}{(\sum_{i=1}^{n} A_i)^2} \tag{4}$$

However, it is easy to show that if $0<A_{min}<A_i<A_{max}$ and $\sigma^2(L_i|X)<\sigma^2_{max}$ for all i with finite boundaries A_{max} and σ^2_{max} , then $\sigma^2(L_P|X)\rightarrow 0$ as $n\rightarrow \infty$ for every given realization of X. For n sufficiently large, the conditional variance tends to zero and the probability for an arbitrary small deviation of L_P from the conditional mean $\mu(L_P|X)$ gets arbitrary small.

Therefore, as a consequence of the law of large numbers, the conditional portfolio loss becomes non-stochastic in a very large, infinitely fine-grained credit portfolio. This is the mathematical formulation of the fact how borrower-specific or unsystematic risk can be eliminated through diversification. The only risk that remains is systematic risk, that is the risk that the actual values of the systematic risk factors $X = (X_1,...,X_k)$ result in a higher or lower value of the conditional mean

$\mu(L_P|X)$. If systematic risk factors are varying, the portfolio loss, considered as a percentage of total exposure, fluctuates respectively.

If some lumpy credit risk remains within the portfolio, the then non-zero conditional variance $\sigma^2(L_P|X)$ is a natural measure for the amount of unsystematic risk inherent to the credit portfolio. The conditional variance will therefore play an prominent role in the formula for the granularity adjustment to be developed later. Note that the conditional variance $\sigma^2(L_P|X)$ depends on the realization of the systematic risk factors. In the given context, the values of $\sigma^2(L_P|X)$ in those scenarios where the realization of the systematic risk factors give rise to high losses are of particular importance.

Two additional remarks concerning the conditional variance can be made. First, as a direct consequence of the so-called law of conditional variance, the average conditional variance over all possible scenarios for the systematic risk factors equals the difference between the variance of L_P and the variance of $\mu(L_P|X)$:

$$\mu[\sigma^2(L_P|X)] = \sigma^2(L_P) - \sigma^2[\mu(L_P|X)] \qquad (5)$$

That is, the expectation of $\sigma^2(L_P|X)$ is that part of the portfolio variance that is caused by unsystematic risk.

Second, the similarities between the conditional variance and the Herfindahl index are obvious. The Herfindahl index

$$H = \frac{\sum_{i=1}^{n} A_i^2}{(\sum_{i=1}^{n} A_i)^2} \qquad (6)$$

is an often used measure to quantify the degree of concentration in credit portfolios. It is proportional to conditional variance $\sigma^2(L_P|X)$ if it is assumed that for each borrower i, the conditional variances $\sigma^2(L_i|X)$ of the individual loan loss variables L_i are the same. This implies that differences regarding the distribution of potential losses between the different borrowers can be neglected. Concentration risks can then only arise from differences regarding the exposure sizes A_i. However, if loans not only differ with respect to exposure sizes, but also with respect to e.g. default probabilities or losses given default, then the Herfindahl index might be a to simple measure of concentration risks.

4. Granularity Adjustment

In calculating *VaR* or *ES* of the portfolio loss in a perfectly diversified credit loan portfolio, the random variable L_P can be replaced by the random variable $\mu(L_P|X)$. If the portfolio is not perfectly diversified, an adjustment for unsystematic risk has to be made. This so called granularity adjustment is the difference of the value of the respective risk measure if it is calculated for $\mu(L_P|X)$ and for L_P, respectively. It can be derived via a sensitivity analysis of *VaR* and *ES*.

4.1 Granularity Adjustment for Value at Risk

$VaR = VaR_{1-\alpha}(L_P)$ is the worst loss which will be only exceed with a small probability α. In case of a continuous probability distribution, it is implicitly defined through:

$$Prob(L_P > VaR_{1-\alpha}(L_P)) = \alpha \tag{7}$$

If one looks at the difference of *VaR* for L_P and for $\mu(L_P|X)$, it turns out that the first order approximation of the error term is zero. To get the formula for the granularity adjustment, it is therefore necessary to also know the second derivative of *VaR*. With the technical details left to the appendix, one finally gets the following closed-end formula:

$$\begin{aligned} VaR_{1-\alpha}(L_P) & \\ &= VaR_{1-\alpha}[\mu(L_P|X) + L_P - \mu(L_P|X)] \\ &\approx VaR_{1-\alpha}[\mu(L_P|X)] \\ &\quad -\frac{1}{2}\left[\frac{\delta\sigma^2[L_P|\mu(L_P|X)=s]}{\delta s} + \sigma^2[L_P|\mu(L_P|X)=s]\frac{\delta ln\, f_{\mu(L_P|X)}(s)}{\delta s}\right]_{s=VaR_{1-\alpha}[\mu(L_P|X)]} \end{aligned} \tag{8}$$

Here, $f_{\mu(L_P|X)}(s)$ denotes the density of the random variable $\mu(L_P|X)$. Note that $\mu(L_P|X)$ is a scalar defined as a function of one or more systematic risk factors $X = (X_1,\ldots,X_k)$. Contrary to the results presented in the literature, this formula for the granularity adjustment is not restricted to the one factor case.

An illustration with a very simple example may be useful. Consider a one-factor model for a completely homogeneous credit portfolio with A_i=1 for all i and

$$L_i = \begin{cases} 1 & \text{with probability } p(X) = X \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

In this case, the conditional default probability $p(X)=X$ is identical to the systematic risk factor, and one has:

$$\mu(L_P | X) = X \tag{10}$$

and

$$\sigma^2 [L_P | \mu(L_P | X) = s] = \frac{s(1-s)}{n} \tag{11}$$

If $x_{1\text{-}\alpha}$ denotes the quantile of the systematic risk factor X, formula (8) then simplifies to

$$VaR(L_P) \approx x_{1-\alpha} - \frac{1-2x_{1-\alpha}}{2n} - \frac{x_{1-\alpha}(1 x_{1-\alpha})}{2n} \frac{\delta ln\, f_X(s)}{\delta s} \Big|_{s=x_{1-\alpha}} \tag{12}$$

Here, the granularity adjustment is inversely proportional to the number of loans n and converges to zero as $n \to \infty$.

It has always been taken for granted in the existing literature that the granularity adjustment is positive. However, if one looks to the analytical formula for the granularity adjustment, it is not immediately clear whether this is indeed always the case. In order to develop a counterexample, assume that in the above example the systematic risk factor X has the following density:

$$f_X(X) = \begin{cases} 750\, x(0{,}2 - x) & if \quad 0 < x < 0{,}2 \\ 0 & otherwise \end{cases} \tag{13}$$

Then

$$\frac{\delta \ln f_X(s)}{\delta s} = \frac{\delta(\ln s + \ln(0{,}2 - s))}{\delta s} = \frac{1}{s} - \frac{1}{0{,}2 - s} \tag{14}$$

It is also easy to check that

$$Prob(X < 0{,}12) = \int_0^{0{,}12} 750\, x(0{,}2 - x)\, dx = 0{,}648 \tag{15}$$

It follows that $VaR_{64{,}8\%}[\mu(L_P|X)] = x_{64{,}8\%} = 12\%$ and:

$$VaR_{64{,}8\%}(L_P) \approx 0{,}12 - \frac{0{,}16}{n} \tag{16}$$

If the bank wants to survive with a probability of 64,80%, the capital charge in a perfectly diversified portfolio would be exactly 12% of total exposure. A not perfectly diversified bank with say n=100 loans would require a slightly lower (!) capital charge of only 11,84%.

From the analytical formula for the granularity adjustment, an explanation is possible how the lack of diversification could, in certain cases, result in a lower *VaR*. First note that with perfect diversification, the bank collapses if $\mu(L_P|X) > VaR_{1-\alpha}(L_P)$ and survives if $\mu(L_P|X) < VaR_{1-\alpha}(L_P)$. If the credit loan portfolio is not perfectly diversified, the bank could also collapse if $\mu(L_P|X) < VaR_{1-\alpha}(L_P)$, and an additional capital buffer is therefore necessary to cover unsystematic risk. However, one should also note that for a not perfectly diversified bank it is also possible to survive even though $\mu(L_P|X) > VaR_{1-\alpha}(L_P)$. In the later case, in which all perfectly diversified banks would collapse, the lack of diversification is obviously an advantage.

Which of these two cases has greater impact depends on the amount of unsystematic risk in the respective scenarios, which is expressed by the value of the conditional variance, and also the occurrence probabilities of these scenarios. If the conditional variance $\sigma^2(L_P|\mu(L_P|X)=s)$ is an increasing function of s, the probability that the bank survives even though the realization of the systematic factors is such that $\mu(L_P|X) > VaR$ is relatively higher than the risk of collapse given a scenario with $\mu(L_P|X) < VaR$. In this case, the first summand $-(1/2)\delta\sigma^2/\delta s$ within the formula for the granularity adjustment is negative.

The second summand $-(1/2)\sigma^2 \, \delta ln(f_\mu)/\delta s$ of the granularity adjustment is positive if the density of the random variable $\mu=\mu(L_P|X)$ slopes downwards in the right tail, which will usually be the case. The occurrence probability of a scenario where the conditional mean is below *VaR* - in which case all perfectly diversified banks would survive - is then higher than the probability of the opposite. However, as the above example shows, there are certain cases where a negative first summand within the granularity adjustment outweighs a positive second summand.

4.2 Granularity Adjustment for Expected Shortfall

ES is an often proposed alternative to *VaR*. It is defined as the average loss on condition that losses are greater or equal than *VaR,* and therefore does not only take into account the probability that losses exceed *VaR*, but also differentiates between small and very large violation of the *VaR*-threshold. This could be motivated by the fact that not only the probability that the bank collapses is of interest to the depositors of the bank, but also whether they will loss everything or only a small amount of money in case that the bank actually collapses. *ES* describes the expected loss in case of a bank collapse[1].

The formal definition of ES with confidence level 1-α is as follows:

$$\begin{aligned} ES_{1-\alpha}(L_P) &= \mu(L_P | L_P > VaR_{1-\alpha}(L_P)) \\ &= \frac{1}{\alpha}\int_{VaR_{1-\alpha}(L_P)}^{\infty} s\, dF_{L_P}(s) \end{aligned} \tag{17}$$

where $F_{L_P}(s)=Prob(L_P<s)$ is the cumulative probability distribution function of the portfolio loss L_P. From the substitution $s=VaR_{1-r}(L_P)$ <=> $F_{L_P}(s)=1-r$ it is obvious that *ES* can also be written as the average *VaR* for all confidence levels above 1-α:[2]

$$ES_{1-\alpha}(L_P)=\frac{1}{\alpha}\int_0^{\alpha} VaR_{1-r}(L_P)\,dr \tag{18}$$

1 However, this implicitly assumes that depositors are risk-neutral or that a pseudo risk-neutral probability distribution for the valuation of state-dependent pay-offs exists.
2 It is assumed that the distribution is continuous. Otherwise, certain additional considerations are necessary, see Pflug (2000), Tasche (2002) for details.

For this reason, ES is sometimes also called Conditional-VaR. Conversely, $VaR_{1-\alpha}$ can be get as the derivative with respect to α of $ES_{1-\alpha}$ times α.

The formula for the granularity adjustment, if VaR is replaced by ES, is as follows:

$$\begin{aligned} ES_{1-\alpha}(L_P) &= ES_{1-\alpha}[\mu(L_P|X)+L_P-\mu(L_P|X)] \\ &\approx ES_{1-\alpha}[\mu(L_P|X)]+\frac{\sigma^2[L_P|\mu(L_P|X)=s]\,f_{\mu(L_P|X)}(s)}{2\alpha}\Big|_{s=VaR_{1-\alpha}[\mu(L_P|X)]} \end{aligned} \tag{19}$$

In case of the simple example which has been used in the previous chapter as an illustration for the granularity adjustment, this now simplifies to:

$$ES_{1-\alpha}(L_P)\approx\int_0^{\alpha} x_{1-r}\,dr+\frac{x_{1-\alpha}(1-x_{1-\alpha})\,f_X(x_{1-\alpha})}{2n\alpha} \tag{20}$$

The granularity adjustment for ES is the product of the conditional variance and the density of the conditional mean divided through by 2α. It can therefore never be negative.

To get an intuition for this, assume 1,000 randomly generated values of the systematic risk factors $X^{(j)}=(X_1^{(j)},\dots,X_k^{(j)})$. In each scenario $X=X^{(j)}$, the conditional expected portfolio loss $\mu(L_P|X^{(j)})$ is calculated and the results are ranked beginning with the highest conditional loss:

$$\mu(L_P|X^{(1)}) > \mu(L_P|X^{(2)}) > \mu(L_P|X^{(3)}) > \tag{21}$$

$ES_{99\%}[\mu(L_P|X)]$ then would be approximately the average of the 10 first results.

In order to compare this with the result then L_P instead of $\mu(L_P|X)$ is taken as the argument of ES, first assume that within each scenario $X=X^{(j)}$ the result for $\mu(L_P|X^{(j)})$ is replaced by say 100 values of

$$L_P=\frac{\sum_{i=1}^{n} A_i L_i(X,\varepsilon_i)}{\sum_{i=1}^{n} A_i} \tag{22}$$

all with $X=X^{(j)}$ but with 100 different, randomly generated values of the unsystematic risk factors $\varepsilon_1,\ldots,\varepsilon_n$. Because for every scenario with $X=X^{(j)}$ the average of these 100 values of L_P roughly equals $\mu(L_P|X^{(j)})$, an approximation of $ES_{99\%}[\mu(L_P|X)]$ is then given as the average of the first 1,000 results out of a total of now 100,000 random realizations of L_P. However, to get *ES* of L_P instead of $\mu(L_P|X)$, such a replacement must be followed by a re-ranking of the results. It is clear that such a re-ranking will give a higher or at best an equal average of the first 1,000 results. It follows from this simple thought experiment that *ES* of $\mu(L_P|X)$ could never overestimate *ES* of L_P. As a consequence, the granularity adjustment for *ES* must be positive.

Conclusion

The granularity adjustment has been developed as a technical tool which accounts for the remaining concentration risks in a not perfectly diversified portfolio. The key idea is instead of calculating systematic and unsystematic risk simultaneously, a two step model is developed with an add-up for unsystematic risk.

It has always been assumed that it goes without saying that such an add-up for unsystematic risk has positive value. However, a counter example was given in this paper if *VaR* is taken as a risk measure. To understand this phenomena, one should note that unsystematic risk could also prevent a bank collapse. Particular in very bad states of nature, some unsystematic volatility within the portfolio could be an advantage. It therefore depends on the value of the conditional variance in such bad states of nature relatively to its value in the good states whether the granularity adjustment will be in fact negative.

The granularity adjustment technique could also be applied if *VaR* is replaced by *ES* as a risk measure. Because *ES* is a coherent measure of risk in the sense of Artzner et al. (1999), the adjustment term will then be always positive. Alongside with the mathematical formulas, an intuitive explanation for this result was also given.

Appendix:

The formulas for the granularity adjustment follow from a sensitivity analysis of *VaR* and *ES*, respectively. In the first step, the partial derivatives of *VaR* and *ES* are proven by the following

1. Lemma: Derivatives of *VaR* and *ES*

With abbreviation $VaR = VaR_{1-\alpha}(Y+hZ)$, $ES = ES_{1-\alpha}(Y+hZ)$ and random variables *Y, Z* and *R,* derivatives of *VaR* and *ES* are given as:

(i) $$\frac{\delta VaR}{\delta h} = \mu[Z|Y+hZ=VaR]$$

(ii) $$\frac{\delta^2 VaR}{\delta h^2} = -\left[\frac{\delta\sigma^2(Z|Y+hZ=s)}{\delta s} + \sigma^2(Z|Y+hZ=s)\frac{\delta ln\, f_{Y+hZ}(s)}{\delta s}\right]_{s=VaR}$$

(iii) $$\frac{\delta ES}{\delta h} = \mu(Z|Y+hZ>VaR)$$

(iv) $$\frac{\delta ES^2}{\delta h^2} = \frac{\sigma^2(Z|Y+hZ=VaR)\, f_{Y+hZ}(VaR)}{\alpha}$$

(iv) is a special case of:

(v) $$\frac{\delta}{\delta h}\mu(R|Y+hZ>VaR) = \frac{Cov(R,Z|Y+hZ=VaR)\, f_{Y+hZ}(VaR)}{\alpha}$$

Proof:

ad (i): $$0 = \frac{\delta}{\delta h} Prob(Y+hZ>VaR)$$

$$= \frac{\delta}{\delta h}\int_{-\infty}^{\infty}\int_{VaR-hz}^{\infty} f(y,z)\, dy\, dz$$

$$= \int_{-\infty}^{\infty}\left(\frac{\delta VaR}{\delta h} - z\right) f(VaR-hz, z)\, dz$$

Because of

$$f(VaR - hz, z) = f_Z(z|Y+hZ=VaR)\ f_{Y+hZ}(VaR)$$

the result for the first derivative of *VaR* follows by dividing through by $f_{Y+hZ}(VaR)$.

ad (v):

$$\begin{aligned}
&\frac{\delta}{\delta h}\mu(R|Y+hZ>VaR) \\
&= \frac{\delta}{\delta h}\int_{-\infty}^{\infty} r\, f_R(r|Y+hZ>VaR)\,dr \\
&= \frac{\delta}{\delta h}\int_{-\infty}^{\infty} r\, \frac{\int_{-\infty}^{\infty}\int_{VaR-hz}^{\infty} f_{R,Y,Z}(r,y,z)\,dydz}{\alpha}\,dr \\
&= \int_{-\infty}^{\infty} r\, \frac{\int_{-\infty}^{\infty} -(\frac{\delta VaR}{\delta h}-z)\, f_{R,Y,Z}(r, VaR-hz, z)\,dz}{\alpha}\,dr \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(z-\frac{\delta VaR}{\delta h})\frac{f_{R,Z}(r,z|Y+hZ=VaR)\,f_{Y+hZ}(VaR)}{\alpha}\,dzdr \\
&= [\mu(RZ|Y+hZ=VaR)-\mu(R|Y+hZ=VaR)\,u(Z|Y+hZ=VaR)]\frac{f_{Y+hZ}(VaR)}{\alpha} \\
&= \frac{Cov(R,Z|Y+hZ=VaR)\,f_{Y+hZ}(VaR)}{\alpha}
\end{aligned}$$

ad (iii): Having proved (v), one could proceed as follows:

$$\begin{aligned}
\frac{\delta}{\delta h}ES(Y+hZ) &= \frac{\delta}{\delta h}\mu(Y+hZ|Y+hZ>VaR) \\
&= \mu(Z|Y+hZ>VaR)+\frac{\delta}{\delta h}\mu(Y+hZ|Y+hZ>VaR) \\
&= \mu(Z|Y+hZ>VaR)+\frac{\delta\mu(Y+hZ|Y+tZ>VaR)}{\delta t}\Big|_{t=h} \\
&= \mu(Z|Y+hZ>VaR)+\frac{Cov(Y+hZ, Z|Y+hZ=VaR)\,f_{Y+hZ}(VaR)}{\alpha} \\
&= \mu(Z|Y+hZ>VaR)
\end{aligned}$$

ad (ii): From equation (18), it follows that

$$VaR_{1-\alpha}(Y+hZ)=\frac{\delta}{\delta\alpha}[\alpha ES_{1-\alpha}(Y+hZ)]$$

Because of $VaR=F_Y^{-1}(1-\alpha)$ one has:

$$\begin{aligned}\frac{\delta^2}{\delta h^2}VaR_{1-\alpha}(Y+hZ)=\frac{\delta^2}{\delta h^2}\frac{\delta}{\delta\alpha}[\alpha ES_{1-\alpha}(Y+hZ)] \\ &= \frac{\delta[\sigma^2(Z|Y+hZ=VaR)f_{Y+hZ}(VaR)]}{\delta\alpha} \\ &= \frac{\delta[\sigma^2(Z|Y+hZ=s)f_{Y+hZ}(s)]}{\delta s}\Big|_{s=Var}\frac{\delta F_Y^{-1}(1-\alpha)}{\delta\alpha} \\ &= \frac{\delta[\sigma^2(Z|Y+hZ=s)f_{Y+hZ}(s)]}{\delta s}\Big|_{s=Var}\frac{-1}{f_Y(VaR)} \\ &= -\Big[\frac{\delta\sigma^2(Z|Y+hZ=s)}{\delta s}+\sigma^2(Z|Y+hZ=s)\frac{\delta ln f_{Y+hZ}(s)}{\delta s}\Big]_{s=VaR}\end{aligned}$$

q.e.d.

2. Granularity Adjustment

First note that for any risk measure $r=VaR$ or $r=ES$:

$$\begin{aligned}r(L_P) &= r[\mu(L_P|X)+L_P-\mu(L_P|X)] \\ &\approx r[\mu(L_P|X)]+\frac{\delta r[\mu(L_P|X)+h(L_P-\mu(L_P|X))]}{\delta h}\Big|_{h=0} \\ &\qquad +\frac{1}{2}\frac{\delta^2 r[\mu(L_P|X)+h(L_P-\mu(L_P|X))]}{\delta h^2}\Big|_{h=0}\end{aligned}$$

Because of the law of iterated expectation, which is also valid for conditional probability measures, it is true that $\mu[\mu(L_P|X)|A] = \mu(L_P|A)$ for $A = \{\mu(L_P|X)=VaR\}$ and $A = \{\mu(L_P|X)>VaR\}$. It follows that the first derivative is zero for $r=VaR$ and $r=ES$:

$$\frac{\delta r[\mu(L_P|X)+h(L_P-\mu(L_P|X))]}{\delta h}\Big|_{h=0} = 0$$

The formulas for the granularity adjustment then are an immediate consequence of the results for the second derivatives of *VaR* and *ES*.

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