Value at Risk, Expected Shortfall, and Marginal Risk Contribution

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1. Introduction

Value at risk (VaR) is today the standard tool in risk management for banks and other financial institutions. It is defined as the worst loss for a given confidence level: For a confidence level of e.g. p=99%, one is 99% certain that at the end of a chosen risk horizon there will be no greater loss than just the VaR. In terms of probability theory, VaR is the 1% quantile (in general the (1-p)% quantile) of the profit and loss distribution.

A simple case is the assumption of a normal distribution, because then VaR is simply a multiple of the standard deviation[1] (e.g. for a confidence level of 99%, VaR is 2.33 standard deviations). In this case, the concept of VaR would not generate any new theoretical problems. VaR would only be a different, less technical form of risk reporting, in which the term „standard deviation“ is replaced by the perhaps easier to understand term „Value at Risk“. However, it is wellkown that the assumption of a normal distribution is questionable for stock market quotations. It is with particular importance for risk management that high losses are far more probable in the stock market than the assumption of a normal distribution would suggest. Also, if VaR is applied to credit risk, it is immediately obvious from the asymmetry of credit risk (small probability of a high loss far below the average outcome) that the loss distribution of a credit loan portfolio cannot be described by a symmetric normal distribution.

Without the assumption of a normal distribution, VaR is a very problematic risk measure. These problems will be illustrated in the next section. Subsequently, I concentrate on one specific issue, namely convexity and sub-additivity of a risk measure. In order to check for convexity, first and second derivatives of VaR are calculated. The same calculations are then repeated for expected shortfall, which is often proposed as an alternative for VaR.

[1] More generally, this holds for all elliptic probability distributions.

2. Value at Risk as a problematic risk measure

For an illustration of the problems of VaR as a risk measure, consider a bank where a VaR-limit (confidence level 99%) of say 50 000 Euro is imposed on a certain trader. The meaning of this is that a loss of more than 50 000 Euro should occur only once in every hundred trading days on average. But because of the very definition of VaR, there is no differentiation between small and very large violations of the 50 000 Euro limit. The eventual loss could be 60 000 Euro as well as 600 000 Euro. Therefore, in particular then VaR is used as a criterion for risk-adjusted compensation, the trader has an incentive to run a strategy which would create an additional profit in most cases, but at the expense of a probability just below 1% of huge losses. For example, the trader could[2]:

• sell options far out the money and earn the respective premiums, where the probability that the option will be exercised is below 1%.
• buy high yield bonds on credit, again on condition that the default probability of the bonds is sufficiently low (such a strategy was reportedly persecuted by Long Term Capital Management, the Hedge Fund which eventually brokes down in 1998)

Of course, in every real-world bank, additional restrictions are probably in place which will prevent the trader from running such strategies. But these additional restrictions are only necessary because of the given deficiencies of VaR. The emerging principal agent problem could be completely avoided if risk-adjusted compensation would be based on a criterion which correctly reflects the riskiness of a portfolio.

Here, it is not my intention to give a systematic overwiew about all problems of VaR[3]. The most important issues are:
• VaR could violate second order stochastic dominance and therefore does not always describe risk aversion in the traditional sense[4].
• VaR is not smooth: Events with a probability just below 1% are not taken into account. This changes immediatly if the probability is exactly or greater than 1%.

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[2] Jaschke (2001).
[3] A good overview is Pflug (2000). An axiomatic approach for so-called coherent risk measures has been developed by Artzner et al. (1999). VaR is not a coherent risk measure.
[4] See for example Guthoff et al. (1998).

• VaR is not always sub-additive: If VaR is calculated for each unit within a bank, the sum of the Values at Risk of each unit could be lower(!) than the VaR for the whole bank. Obviously, this contradicts the idea of diversification, because risk could than be reduced by running each unit seperately. (These centrifugal forces are presumably not in the interest of the top management). The lack of sub-additivity makes VaR a problematic criterion for portfolio optimization, the internal allocation of capital, and for the design of RAROC-type risk-adjusted compensation schemes.

For an example which illustrates why VaR is not always sub-additive, consider a loan with a default probability below 1%. For a portfolio which contains only one loan, VaR is obviously zero. (There will be no loss with a probability of at least 99%). If sufficiently many of such loans are pooled within the same portfolio, almost surely some of the loans will default, resulting in a VaR now greater than zero.

3. Derivatives of Value at Risk and Expected Shortfall

3.1 Preliminary remarks

In order to get a better understanding of the problems of VaR mentioned above, the marginal behaviour of VaR if a new position is added to the portfolio could be studied. In practice, marginal risk contributions are often deduced from the contribution of the new position to the standard deviation of the portfolio. However, without the assumption of a normal distribution, there is no close relationship between standard deviation and VaR. So what we want to get is a general formula for marginal risk contributions which does not rely on specific assumptions about the profit and loss distribution.

Suppose that the value of the actual portfolio is described by a random variable X and that a fraction a of another random variable Y is added to that portfolio. It is then possible to calculate the derivatives of a certain risk measure with respect to a. Of particular interest is the second derivative which must be positive for a convex risk measure which fulfils the property of sub-additivity. The standard deviation for example is a convex risk measure as can be seen from the curve shape of the efficient frontier in the usual risk-return chart. We conclude that the standard deviation is also sub-additive[5]:

$$\frac{\partial^2 \sigma(X + aY)}{\partial a^2} > 0$$

$$\Rightarrow \sigma(\frac{X+Y}{2}) = \sigma(X + \frac{1}{2}(Y - X)) = f(\frac{1}{2}) < \frac{f(0) + f(1)}{2} = \frac{\sigma(X) + \sigma(Y)}{2}$$

3.2 First and second derivative of Value at Risk

Let us now replace the standard deviation by VaR as an alternative measure of risk. Assume that X, Y are continuously distributed random variables (where f_X denotes the density of X) and define $VaR_p(X + aY)$ as function of a implicitly by $Prob(-X - aY \le VaR_p(X + aY)) = p = const.$. We then have a nice result: The derivative of the VaR is the conditional expectation of the marginal position, on condition that the actual value of the portfolio X and VaR are exactly identical[6]:

$$\left.\frac{\partial VaR_p(X + aY)}{\partial a}\right|_{a=0} = \mu(-Y \mid -X = VaR_p(X))$$

The intuiton behind this result is as follows: If X>*VaR(X)* (the actual loss is already greater than VaR) or if *X<VaR(X)* (there is a remainig buffer), adding a sufficiently small fraction of a new risk would not change the outcome. Therefore, it seems plausible that the risk contribution is the average value for all critical cases with *X=VaR(X)*.

For the second derivative we get the following expression[7]:

[5] Note that the standard deviation is also linear homogeneous, i.e. $\sigma(aX) = a\sigma(X)$. The same holds for VaR and expected shortfall.
[6] Gourieroux et al. (2000), Tasche (1999).
[7] Gourieroux et al. (2000).

$$\frac{\partial^2 VaR_p(X+aY)}{\partial a^2}\Big|_{a=0} = \left[\frac{\partial\sigma^2(Y|X=x)}{\partial x} + \sigma^2(Y|X=x)\ \frac{\partial \ln f_X(x)}{\partial x}\right]_{x=-VaR_p(X)}$$

This is the sum of two terms. The sign of the second term is positive if the density slopes upwards in the left tail. This will usually be the case (if the distribution is unimodal). Unclear is the sign of the first term. To get an intuition, note that the new position which is added to the portfolio could also lift the value of the portfolio above the VaR-threshold if a violation of that threshold would otherwise occur. If $\partial\sigma^2(Y|x)/\partial x$ is negative (the variance is a decreasing function of *x*), the chance that the new position prevents a violation of the VaR-threshold is greater than the corresponding risk that a violation of the VaR-threshold is triggered by the new position. This explains why, in a second order approximation, the contribution to VaR could be lower than the respective conditional mean. The bottom line is that we cannot be sure that the second derivative is always positive.

3.3 First and second derivative of Expected Shortfall

Expected shortfall (ES) is defined as the average of all losses which are greater or equal than VaR, i.e. the average loss in the worst *(1-p)%* cases. For a continuous distribution, ES is the same as Conditional VaR, where Conditional VaR is defined as the average VaR for all confidence levels above *p*:[8]

$$ES_p(X) = \mu(-X|-X \geq VaR_p(X)) = \frac{1}{1-p}\int_p^1 VaR_s(X)\ ds$$

It follows that VaR is the negative derivative with respect to *p* of ES times *1-p*. (the following results therefore contain the previous ones as special cases).

The first derivative of ES is the conditional mean of the marginal position, now on condition that the portfolio value is below VaR[9]:

[8] For a proof, simply substitute $z = VaR_s(X) \Leftrightarrow s = Prob(-X \leq z)$.
[9] Tasche (1999).

$$\frac{\partial ES_p(X+aY)}{\partial a}\Big|_{a=0} = \mu(-Y \mid -X \geq VaR_p(X))$$

At this point, let us try to get an intuition for marginal VaR and marginal ES. Consider a Monte Carlo simulation with 1000 iterations. The results are ranked from the worst loss to the highest gain, so that VaR for a confidence level of 99% is the outcome in the 10th worst scenario. The portfolio value in the 10th worst scenario is the sum of the values of each individual position. Accordingly, the values of the individual positions in that particular scenario are the VaR-contributions or an estimation of the derivative of VaR. This illustrates why the derivative of VaR is the conditional mean, on condition that portfolio value and VaR are identical. However, the accuracy of a so calculated predictor of the conditional mean is very doubtful, because a completely new Monte Carlo simulation could deliver very different values for the individual positions. Only on the aggregate level for the portfolio as a whole, such random errors due to simulation will be largely eliminated.

Now consider ES, which is the average portfolio value in the 10 worst scenarios. The contributions to ES are the average values of the individual positions in the 10 worst scenarios, in accordance with our formal result for the derivative of ES. In addition, because the calculation of marginal risk contributions is now based on the outcome of 10 scenarios (rather than only one), they are presumably less subject to simulation errors. This advantage would be even greater for a simulation with 5000 or 10 000 iterations.

But the advantage of ES lies not only in the calculation of the marginal risk contributions. It also has the advantage of being convex and sub-additive. This follows directly from the following expression for the second derivative (this relatively simply expression is, to my best knowledge, a completely new result):

$$\frac{\partial^2 ES_p(X+aY)}{\partial a^2}\Big|_{a=0} = \frac{1}{1-p}\left[\sigma^2(Y \mid X=x)\; f_X(x)\right]_{x=-VaR_p(X)} > 0$$

4. Outlook

As often, there are bad news and good news. The bad news are that VaR, despite it is widely used in practice, is not an adequate measure of risk. In particular, VaR is not always sub-additive and therefore an inappropriate tool for risk-adjusted performance measurement and the internal allocation of capital. The good news are that an alternative to VaR is given by ES, which not only fulfils the property of sub-additivity, but also makes it easier to calculate marginal risk contributions in practice.

Many open questions remain. I see mainly two topics for future research. First, we have always assumed that the random variables are continuously distributed. However, the above results could be wrong for random variables with a discrete probability distribution. So the question arises what can be said about marginal risk contributions if random variables are not continuously distributed. This is an important issue because all real-world probability distributions are in fact discrete distributions (Consider credit risk as an important example).

The second question relates to the interpretation of a risk measure. VaR is the amount of equity capital which is needed so that the confidence level is the probability that insolvency will not occur. ES however has at first sight no such obvious interpretation. We might see ES as the average loss of the creditors of the bank in case of a default[10]. However, this either presumes risk neutrality or the actual probability distribution must be replaced by a pseudo risk neutral distribution. The question is then how to obtain the risk neutral distribution in practice.

[10] If a new position is added to the portfolio, it then has to be determined how much additional equity must be hold so that expected shortfall remains constant.

<u>Appendix</u>

(i) $$\frac{\partial VaR_p(X+aY)}{\partial a} = \mu(-Y \,|\, -X - aY = VaR_p(X+aY))$$

(ii) $$\frac{\partial^2 VaR_p(X+aY)}{\partial a^2} =$$

$$\left[\frac{\partial \sigma^2(Y \,|\, X+aY=s)}{\partial s} + \sigma^2(Y \,|\, X+aY=s)\ \frac{\partial ln f_{X+aY}(s)}{\partial s}\right]_{s=-VaR_p(X)}$$

(iii) $$\frac{\partial ES_p(X+aY)}{\partial a} = \frac{\partial}{\partial a}\mu(-X-aY \,|\, -X-aY \geq VaR_p(X+aY))$$

$$= \mu(-Y \,|\, -X-aY \geq VaR_p(X+aY))$$

(iv) $$\frac{\partial^2 ES_p(X+aY)}{\partial a^2} = \frac{\partial}{\partial a}\mu(-Y \,|\, -X-aY \geq VaR_p(X+aY))$$

$$= \left[\frac{\sigma^2(Y \,|\, X+aY=s)\, f_{X+aY}(s)}{1-p}\right]_{s=-VaR_p(X+aY)}$$

Proof:

ad (i):

First note that the formula for the conditional density is given by:

$$f_Y(y \,|\, X+aY=-VaR_p(X+aY)) = \frac{f(-VaR_p(X+aY)-ay, y)}{f_{X+aY}(-VaR_p(X+aY))}$$

Then:

$$0 = \frac{\partial}{\partial a} \Pr ob(-X - aY \le VaR_p(X + aY))$$

$$= \frac{\partial}{\partial a} \int_{-\infty}^{+\infty} \int_{-VaR_p(X+aY)-ay}^{+\infty} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} \Big(\frac{\partial VaR_p(X + aY)}{\partial a} + y \Big)\, f(-VaR_p(X + aY) - ay, y)\, dy$$

$$= \left[\Big(\frac{\partial VaR_p(X + aY)}{\partial a} + \int_{-\infty}^{+\infty} y\ f_Y(y \,|\, X + aY = s)\, dy \Big)\ \ f_{X+aY}(s) \right]_{s = -VaR_p(X+aY)}$$

$$= \frac{\partial VaR_p(X + aY)}{\partial a} \ - \ \mu(-Y \,|\, -X - aY = VaR_p(X + aY))$$

ad (iv):

$$\frac{\partial}{\partial a} \mu(-Y \,|\, -X - aY \ge VaR_p(X + aY))$$

$$= \frac{\partial}{\partial a} \int_{-\infty}^{+\infty} -y\ f_Y(y \,|\, -X - aY \ge VaR_p(X + aY))\, dy$$

$$= \frac{\partial}{\partial a} \int_{-\infty}^{+\infty} -y \, \frac{\int_{-\infty}^{-VaR_p(X+aY)-ay} f(x,y)\, dx}{1-p}\, dy$$

$$= \int_{-\infty}^{+\infty} y\ \Big(\frac{\partial VaR_p(X + aY)}{\partial a} + y \Big)\ \frac{f(-VaR_p(X + aY) - ay, y)}{1-p}\ dy$$

$$= \left[\int_{-\infty}^{+\infty} (y^2 - y\, \mu(Y \,|\, X + aY = s)) \frac{f_Y(y \,|\, X + aY = s)\ f_{X+aY}(s)}{1-p}\ dy \right]_{s = -VaR_p(X+aY)}$$

$$= \left[\frac{\sigma^2(Y \,|\, X + aY = s)\, f_{X+aY}(s)}{1-p} \right]_{s = -VaR_p(X+aY)}$$

ad(iii):

$$\frac{\partial}{\partial a}\mu(-X-aY\,|-X-aY\geq VaR_p(X+aY))\Big|_{a=0}$$

$$=\frac{\partial}{\partial a}(\mu(-X\,|-X-aY\geq VaR_p(X+aY))+a\mu(-Y\,|-X-aY\geq VaR_p(X+aY)))\Big|_{a=0}$$

$$=\left[\frac{\sigma^2(X\,|X=x)\quad f_X(x)}{1-p}\right]_{x=-VaR_p(X)} + \mu(-Y\,|-X\geq VaR_p(X)) + 0 \text{ (because of (iv))}$$

$$= \mu(-Y\,|-X\geq VaR_p(X))$$

To get the general result for $a\neq 0$, simply replace X by $X+aY$.

ad (ii):

With the relationship between VaR and ES and (iv), one gets:

$$\frac{\partial^2 VaR_p(X+aY)}{\partial a^2}\Big|_{a=0}$$

$$= -\frac{\partial}{\partial p}\frac{\partial^2((1-p)ES_p(X+aY))}{\partial a^2}\Big|_{a=0}$$

$$= -\frac{\partial}{\partial p}(\sigma^2(Y\,|X=-VaR_p(X))\quad f_X(-VaR_p(X)))$$

$$=\left[\frac{\partial\sigma^2(Y\,|X=x)}{\partial x} + \sigma^2(Y\,|X=x)\ \frac{\partial ln f_X(x)}{\partial x}\right]_{x=-VaR_p(X)}$$

The last step follows because of $VaR_p(X)=-F_X^{-1}(1-p)$, where F_X^{-1} is the cumulative probability distribution function of X, and $\frac{\partial F_X^{-1}(1-p)}{\partial p} = \frac{-1}{f_X(F_X^{-1}(1-p))}$. To get the general result for $a\neq 0$, again replace X by $X+aY$.

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